LIPSCHITZIAN PROPERTIES AND STABILITY OF A CLASS OF FIRST-ORDER STOCHASTIC DOMINANCE CONSTRAINTS

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Abstract. Considering first-order stochastic dominance constraints for random variables arising as optimal values of stochastic programs with linear recourse, verifiable sufficient conditions for metric regularity are presented. A growth condition developed in [22] has a crucial role in the analysis of the present paper. Implications regarding stability and sensitivity of optimal values and optimal solutions of stochastic optimization problems involving the dominance constraints considered conclude the paper.

Key words. Stochastic programming, stochastic dominance, Lipschitzian properties.

AMS subject classifications. 90C15, 90C11, 60E15.

1. Introduction.
Since their mobilization for optimization under stochastic uncertainty in the seminal paper [9], stochastic programming problems with dominance constraints have seen a rapid increase in attention among researchers. In the meantime, research on stochastic dominance in stochastic programming has developed into different directions. Topics include basic structural properties of general models, [9, 10, 13], solution methods, [12, 24, 28, 31, 38], practical applications, [11, 15, 16], and stability analysis, [6, 8, 26]. A specific line of research concerns dominance constraints for random variables arising in two-stage stochastic programming, [4, 5, 7, 14, 17, 18, 19]. In the present paper, we address a class of problems of this type. We start out from random two-stage optimization problems

\[ \min_{x,y} \{ c^\top x + q^\top y : Tx + Wy = z(\omega), x \in X, y \in \mathbb{R}^{m_2} \} \quad (1.1) \]

where \( x \in X \subseteq \mathbb{R}^{m_1} \) has to be selected in here-and-now fashion, i.e., without knowing the realization of \( z(\omega) \). The latter is assumed a random vector on some probability space \((\Omega, \mathcal{A}, \mathbb{P})\) with values in \( \mathbb{R}^s \) and distribution not depending on \( x \). Here, \( X \) is a non-empty polyhedron, and all remaining variables and data have conformable dimension.

After observation of \( z(\omega) \), the decision on \( y = y(x, \omega) \) is taken best possible, namely as an optimal solution to the remaining optimization problem in the \( y \)-variables after decision on \( x \) and observation of \( z(\omega) \).

Proceeding in this way, brings the following family of random variables \( \{ f_x(\omega) \}_{x \in X} \) to the fore:

\[ f_x(\omega) := f(x, z(\omega)) = c^\top x + \min_y \{ q^\top y : Wy = z(\omega) - Tx, y \in \mathbb{R}^{m_2} \}, \quad x \in X. \quad (1.2) \]

The function \( f(x, z) \) is well-defined on \( X \times \mathbb{R}^s \) under the following assumptions:

\( \textbf{(A1)} \) (Complete Recourse) \( W(\mathbb{R}^{m_2}) = \mathbb{R}^s \).

\( \textbf{(A2)} \) (Sufficiently expensive recourse) \( M_D := \{ u \in \mathbb{R}^s : W^\top u \leq q \} \neq \emptyset \).

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These assumptions further imply that $M_D$ is bounded, hence the convex hull of finitely many vertices $d_1, \ldots, d_N \in \mathbb{R}^s$, and that the function
\[ \phi(t) := \max \{ t^T u : W^T u \leq q \} = \max_{i=1,\ldots,N} d_i^T t \] (1.3)
is finite, convex, and thus continuous on $\mathbb{R}^s$.

The random variables $f_x(\omega)$ give rise to various formulations of stochastic programs, see [3, 39, 41]. Taking their mean-values yields the classical risk neutral two-stage linear stochastic program with recourse. Forming a weighted sum of the mean and a scalar quantity expressing some perception of risk (risk measure) leads to mean-risk models. In the present paper we do not pursue any further these setups of risk aversion in the objective. Rather we focus on risk aversion in the constraints by means of stochastic orders, i.e., partial orders of random variables as discussed in detail in [30, 40].

We study optimization problems whose constraint sets are determined by what is called the “usual stochastic order” in the literature: A real-valued random variable $X(\omega)$ is called stochastically smaller than a real-valued random variable $Y(\omega)$ if and only if
\[ \mathbb{E}[h(X)] \leq \mathbb{E}[h(Y)] \]
for all nondecreasing functions $h : \mathbb{R} \to \mathbb{R}$ for which both expectations exist. In this case we write $X \preceq_1 Y$ and say $X$ dominates $Y$ to first order. This relation is equivalent to
\[ \mathbb{P}(\{\omega | X(\omega) \leq \eta\}) \geq \mathbb{P}(\{\omega | Y(\omega) \leq \eta\}) \text{ for all } \eta \in \mathbb{R}, \] (1.4)
see for instance [30, 40].

Back at stochastic programming, we fix a real-valued random variable $d(\omega)$ on $(\Omega, \mathcal{A}, \mathbb{P})$ as benchmark cost profile. Considering the random variables $f_x(\omega), x \in X$ from (1.2) and rather aiming at selecting acceptable ones than finding maximal or minimal ones, we say that $f_x$ is acceptable if it dominates the benchmark $d$ to first order. Passing to the image measures $\mu := \mathbb{P} \circ z^{-1} \in \mathcal{P}(\mathbb{R}^s)$ and $\nu := \mathbb{P} \circ d^{-1} \in \mathcal{P}(\mathbb{R})$, with $\mathcal{P}(\mathbb{R}^s), \mathcal{P}(\mathbb{R})$ denoting the sets of all Borel probability measures on the respective Euclidean spaces, we formulate the set-valued mapping
\[ C : \mathcal{P}(\mathbb{R}^s) \to 2^{\mathbb{R}^{m_1}}, \quad C(\mu) := \{x \in X | f(x, z) \preceq_1 d\}. \]

With some disutility function $g : \mathbb{R}^{m_1} \to \mathbb{R}$ we obtain stochastic programs
\[ \mathbb{P}(\mu) \min \{g(x) | x \in C(\mu)\} \]
depending on a parameter $\mu \in \mathcal{P}(\mathbb{R}^s)$. As a prerequisite for subsequent investigations, we equip $\mathcal{P}(\mathbb{R}^s)$ with weak convergence of probability measures ([2]): A sequence $\{\mu_n\}$ in $\mathcal{P}(\mathbb{R}^s)$ is said to converge weakly to $\mu \in \mathcal{P}(\mathbb{R}^s)$, written $\mu_n \xrightarrow{w} \mu$, if for any bounded continuous function $h : \mathbb{R}^s \to \mathbb{R}$ it holds $\int_{\mathbb{R}^s} h(z) \mu_n(dz) \to \int_{\mathbb{R}^s} h(z) \mu(dz)$ as $n \to \infty$. Studying the stability of stochastic programs with respect to perturbations of the underlying probability distribution is mainly motivated by this distribution often resulting from a more or less subjective choice or arising from approximation. In both situations it is desired that small parameter variations cause only small changes
in the optimal value and the set of optimal solutions. Weak convergence of probability measures then is a sufficiently specific selection allowing for substantial results to hold. On the other hand, it is sufficiently general to include important special cases.

As a general reference to stability and sensitivity in stochastic programs we suggest [36]. For dominance constrained stochastic programs, research on this topic is more recent, addressing more general models in [6, 8, 26] and with accent on dominance constraints induced by two-stage problems in [18, 19]. In [19] the parametric program \( P(\mu) \) has been studied in the more general setting with allowing for integer components in \( x \) and \( y \). With (A1), adapted to the mixed-integer setting, maintaining (A2) and assuming rational data, it has been shown that the set-valued mapping \( C \) is a closed multifunction on \( \mathcal{P}(\mathbb{R}^2) \), i.e., for arbitrary \( \mu \in \mathcal{P}(\mathbb{R}^2) \) and sequences \( \mu_n \in \mathcal{P}(\mathbb{R}^2), x_n \in C(\mu_n) \) with \( \mu_n \xrightarrow{w} \mu \) and \( x_n \rightarrow x \) it follows that \( x \in C(\mu) \).

If, in addition, the set \( X \) is compact, and \( g \) a lower semicontinuous function, then the mapping assigning to every \( \mu \in \mathcal{P}(\mathbb{R}^2) \) the global optimal value of \( P(\mu) \) is lower semicontinuous as well.

In the present paper, our focus is on verifyable sufficient conditions finally leading to Lipschitzian properties of multifunctions closely related to \( C \). In [22], the authors demonstrated the crucial role of a growth condition on the functions belonging to active constraints when heading for Lipschitzian properties of probabilistic constraints. Notice that, due to (1.4), the multifunction \( C \) is essentially given by probability inequalities, very much in common with classical chance constraints [33]. Therefore, verification of the growth condition from [22] for inequality constraints related to first-order stochastic dominance is the central theme of the present paper.

Recall that (1.4) is an equivalent characterization of first-order dominance in terms of a continuum of constraints, indexed by \( \eta \in \mathbb{R} \). Therefore, our subsequent analysis proceeds in two essential steps. In Section 2 we study (1.4) for fixed \( \eta \), and in Section 3 we establish verifyable sufficient conditions guaranteeing the desired growth uniformly for subsets \( A \subseteq \mathbb{R} \). In Section 4 we present implications for metric regularity and Lipschitzian properties of the constraints. This, in turn, leads us to stability under perturbations of \( \mu \) of optimal values and sets of optimal solutions to optimization problems \( P(\mu) \) with first-order stochastic dominance constraints. Let us also point to Chapter 9 of [35] highlighting the importance of the Lipschitz properties established and offering points of departure for further investigation, e.g., by the equivalences to metric regularity established in Theorem 9.43 there.

2. Sufficient conditions for local linear growth for a fixed level \( \eta \).

Invoking characterization (1.4) we have

\[
C(\mu) = \{ x \in X \mid \mu\{ \{ z \in \mathbb{R}^n \mid c^T x + \phi(z - T x) \leq \eta \} \geq \nu_{[\mathbb{R} \leq \eta]} \forall \eta \in \mathbb{R} \} \\
= \{ x \in X \mid F_\eta(x) \geq 0 \forall \eta \in \mathbb{R} \},
\]

where we assume (A1) and (A2) and define \( F_\eta : \mathbb{R}^{m+1} \rightarrow \mathbb{R} \) by

\[
F_\eta(x) := \mu\{ \{ z \in \mathbb{R}^n \mid c^T x + \phi(z - T x) \leq \eta \} \} - \nu_{[\mathbb{R} \leq \eta]} \]

\[
(1.3) \Rightarrow \mu\{ \{ z \in \mathbb{R}^n \mid c^T x + \max_{i=1, \ldots, N} d_i^T (z - T x) \leq \eta \} \} - \nu_{[\mathbb{R} \leq \eta]} \\
= \mu\{ \{ z \in \mathbb{R}^n \mid d_i^T z \leq \eta + (d_i^T T - c^T)x \forall i = 1, \ldots, N \} \} - \nu_{[\mathbb{R} \leq \eta]}.
\]
Write
\[ 1 := \left( \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right) \in \mathbb{R}^n, \quad D := \left( \begin{array}{c} d_1^T \\ \vdots \\ d_k^T \end{array} \right) \in \mathbb{R}^{n \times s}, \quad C := \left( \begin{array}{c} c_1^T \\ \vdots \\ c_s^T \end{array} \right) \in \mathbb{R}^{n \times m_1} \]
and denote by \( Q_\eta : \mathbb{R}^{m_1} \rightarrow 2^{\mathbb{R}^s} \) the set-valued mapping given by
\[ Q_\eta(x) := \{ z \in \mathbb{R}^s \mid Dz \leq \eta 1 + (DT - C)x \}. \]
\( F_\eta \) can then be represented as \( F_\eta(x) = \mu[Q_\eta(x)] - \nu[\mathbb{R}^{n_0}] \).

**Remark 1.** Since \( Q_\eta(x) \) is a polyhedron, it is in particular closed and therefore a Borel set. \( \mu \in \mathcal{P}(\mathbb{R}^s) \), consequently \( Q_\eta(x) \) is \( \mu \)-measurable for every \( x \in \mathbb{R}^{m_1} \) and \( \eta \in \mathbb{R} \).

Throughout the rest of this section we consider \( \eta \in \mathbb{R} \) to be fixed and study sufficient conditions for the function \( F_\eta \) to be **growing** in the sense of [22]:

**Definition 2.1.** Let \( F = (F_1, \ldots, F_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a mapping, \( K \subseteq \mathbb{R}^n \) a closed set and \( x_0 \in K \) such that \( F(x_0) \geq 0 \). Furthermore, set
\[
I(x_0) := \{ i \in \{1, \ldots, m\} \mid F_i(x_0) = 0 \}
\]
and
\[
J(x_0) := \{ j \in \{1, \ldots, m\} \mid F_j \text{ is not continuous at } x_0 \}.
\]
We say that \( F \) is **growing** at \( x_0 \) with respect to \( K \) if and only if
1. \( F_i \) is upper semicontinuous in a neighborhood of \( x_0 \) for \( i \in \{1, \ldots, m\} \) and
2. there exist constants \( r > 0 \) and \( p > 0 \) such that for all \( x \in K \cap B_r(x_0) \) and all \( \epsilon > 0 \) there exists a point \( \hat{x} \in K \cap B_{\epsilon}(x) \) with
\[
F_i(\hat{x}) > F_i(x) + p \| \hat{x} - x \| \quad \forall i \in I(x_0) \cup J(x_0).
\]

The following theorem illustrates the relationship between the above growth condition and metric regularity (see Lemma 2 and Lemma 3 in [22]):

**Theorem 2.2.** Let \( F : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a mapping, \( K \subseteq \mathbb{R}^n \) a closed set and \( x_0 \in K \) such that \( F(x_0) \geq 0 \).
1. If \( F \) is growing at \( x_0 \) with respect to \( K \), then the associated set-valued mapping
\[
\Gamma_F : \mathbb{R}^{m_1} \rightarrow 2^{\mathbb{R}^s}
\]
given by
\[
\Gamma_F(x) = \begin{cases} 
\{ F(x) \} \oplus \mathbb{R}^n_{\geq 0}, & \text{if } x \in K \\
\emptyset, & \text{else}
\end{cases}
\]
is metrically regular at \((x_0, 0)\) with respect to \( K \), i.e. there exist constants \( \epsilon > 0 \) and \( \alpha > 0 \) such that for all \( x \in K \cap B_{\epsilon}(x_0) \) and \( l \in B_{\alpha}(0) \) the inequality
\[
\operatorname{dist}_{PS}(x, \Gamma_F^{-1}(l)) \leq \alpha \operatorname{dist}_{PS}(l, \Gamma_F(x))
\]
holds true. Here, \( \oplus \) denotes the Minkowski sum and \( \operatorname{dist}_{PS}(\cdot, \cdot) \) is the point-to-set distance.
2. If \( F \) is continuous and \( \Gamma_F \) is metrically regular at \((x_0, 0)\), then \( F \) is growing at \( x_0 \) with respect to \( K \).
We consider the case where $F = F_\eta$ and hence $m = 1$. First, we use the following result from [37] to show that $F_\eta$ is upper semicontinuous:

**Theorem 2.3.** Let $Y \subseteq \mathbb{R}^m_1$ be a nonempty closed set, $\sigma \in \mathcal{P}(\mathbb{R}^s)$ a Borel probability measure, $H : \mathbb{R}^m_1 \to 2^{\mathbb{R}^s}$ a set-valued mapping having a closed graph and $\mathcal{B} \subseteq \mathbb{B}^s$ a determining set of Borel sets such that $\{H(y) \mid y \in Y\} \subseteq \mathcal{B}$. Then for every $u \in \mathbb{R}$ the graph of the set-valued mapping $R_u : \mathcal{P}(\mathbb{R}^s) \to 2^{\mathbb{R}^m_1}$ given by $R_u(\sigma) := \{y \in Y \mid \sigma[H(y)] \geq u\}$ is closed with respect to the $\mathcal{B}$-discrepancy distance $\mathbf{dist}_\mathcal{B}$ on $\mathcal{P}(\mathbb{R}^s)$, which is given by $\mathbf{dist}_\mathcal{B}(\sigma_1, \sigma_2) := \sup_{B \in \mathcal{B}} ||\sigma_1[B] - \sigma_2[B]||$.

Now the following lemma can be easily obtained:

**Lemma 2.4.** Assume (A1), (A2). Then $F_\eta : \mathbb{R}^m_1 \to \mathbb{R}$ is upper semicontinuous on $\mathbb{R}^m_1$.

**Proof.** $F_\eta$ is upper semicontinuous if and only if for every $\alpha \in \mathbb{R}$ the superlevel set $\text{lev}_{\geq \alpha} F_\eta$ is closed. Set $Y := \mathbb{R}^m_1$. In view of the notation from theorem 2.3 it holds true that $\text{lev}_{\geq \alpha} F_\eta = \{x \in \mathbb{R}^m_1 \mid F_\eta(x) \geq \alpha\} = R_{\alpha+\nu_{\mathcal{B}_\mathcal{L}}}(\mu)$.

$\mathcal{B} := \mathcal{B}^s$ is obviously determining and since $Q_\eta(x) \in \mathcal{B}^s$ for every $x \in \mathbb{R}^m_1$, we have $\{Q_\eta(x) \mid x \in \mathbb{R}^m_1\} \subseteq \mathcal{B}$. We show that the set-valued mapping $Q_\eta : \mathbb{R}^m_1 \to 2^{\mathbb{R}^s}$ has a closed graph: Consider a sequence $\{(x_j, z_j)\}_{j \in \mathbb{N}} \subseteq \text{gph} Q_\eta = \{(x, z) \in \mathbb{R}^m_1 \times \mathbb{R}^s \mid z \in Q_\eta(x)\}$ that converges to $(x, z) \in \mathbb{R}^m_1 \times \mathbb{R}^s$. Since $(x_j, z_j) \in \text{gph} Q_\eta$, we have

$$c^T x_j + \phi(z_j - T x_j) \leq \eta \quad \forall j \in \mathbb{N}.$$  

By (A1) and (A2) $\phi$ is continuous, consequently

$$c^T x_j + \phi(z_j - T x_j) \to c^T x + \phi(z - T x) \quad (j \to \infty).$$

This yields $c^T x + \phi(z - T x) \leq \eta$, implying $(x, z) \in \text{gph} Q_\eta$. Hence, $\text{gph} Q_\eta$ is closed.

Invoking theorem 2.3 we obtain that $\text{gph} R_u$ is closed for all $u \in \mathbb{R}$. In particular, $\text{lev}_{\geq \alpha} F_\eta = R_{\alpha+\nu_{\mathcal{B}_\mathcal{L}}}(\mu)$ is closed for arbitrary $\alpha \in \mathbb{R}$. $\square$

We apply a tightened version of the second part of definition 2.1 to $F_\eta$, where we demand local linear growth with respect to the polyhedron $X$ even if $F_\eta(x_0) \neq 0$ and $F_\eta$ is continuous at $x_0$:

**Condition 1.** There exist constants $r = r(x_0, \eta) > 0$ and $p = p(x_0, \eta) > 0$ such that for all $x \in X \cap B_r(x_0)$ and $\epsilon > 0$ there exists a point $\hat{x} \in X \cap B_\epsilon(x)$ with

$$F_\eta(\hat{x}) > F_\eta(x) + p \|\hat{x} - x\|$$

$$\Leftrightarrow \mu(Q_\eta(\hat{x})) > \mu(Q_\eta(x)) + p \|\hat{x} - x\|.$$  

(2.2)

We will work with the following assumptions:

**G1** All vertices of $M_D$ are different from $0 \in \mathbb{R}^s$.

**L1** The polyhedron $Q_\eta(x_0)$ has full dimension.
Remark 2. (G1) is a global assumption in that it is either true for all or for none of the \( x_0 \in X \). In the same sense (L1) can be considered as a local assumption, since it involves the concrete point \( x_0 \).

Remark 3. (G1) holds true if and only if every row of the matrix \( D \) contains an entry different from 0. Together (G1) and (L1) imply that there exists a point \( z_\leq \in \mathbb{R}^s \) such that \( Dz_\leq < \eta \mathbf{1} + (DT - C)x_0 \) and hence a constant \( \epsilon > 0 \) such that

\[
Q_\eta(x) \neq \emptyset \quad \forall x \in B_\epsilon(x_0).
\] (2.3)

The following example shows what can happen if (G1) is not fulfilled:

Example 1. Recall the basic problem

\[
\min \{ c^T x + q^T y \mid Tx + Wy = z(\omega), \ x \in X, \ y \in \mathbb{R}_+^{m_2} \}
\]

and consider the case where \( c, d \in \mathbb{R} \) are arbitrary, \( X = \mathbb{R} \), \( m_2 = 4 \), \( q = e_1 \in \mathbb{R}^4 \), \( T = (c,d)^T \in \mathbb{R}^2 \) and \( W = (I_2, -I_2) \in \mathbb{R}^{2 \times 4} \) with \( e_1 \) denoting the first unit vector and \( I_2 \) the \((2 \times 2)\)-unit matrix. (A1) holds true and since

\[
M_D = \{ u \in \mathbb{R}^2 \mid W^T u \leq q \} = \{(u_1, u_2)^T \in \mathbb{R}^2 \mid 0 \leq u_1 \leq 1, \ u_2 = 0 \} \neq \emptyset,
\]

(A2) is also fulfilled. \( M_D \) has the two vertices \((1,0)^T\) and \((0,0)^T\), consequently for arbitrary \( x \in \mathbb{R} \) we have

\[
Q_\eta(x) = \{ z \in \mathbb{R}^2 \mid Dz \leq \eta \mathbf{1} + (DT - C)x \} = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^2 \mid \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \leq \begin{pmatrix} \eta \\ \eta - cx \end{pmatrix} \right\}
\]

\[
= \begin{cases} \mathbb{R} \leq \eta \times \mathbb{R} \times \mathbb{R}, & \text{if } \eta \geq cx \\ \emptyset, & \text{else} \end{cases}
\]

For every \( x_0 \in \mathbb{R} \) and every \( \epsilon > 0 \) the interval \([x_0 - \epsilon, x_0 + \epsilon] \subseteq \mathbb{R} \) contains a nontrivial subinterval on which \( Q_\eta \) is constant. Hence condition 1 cannot be fulfilled.

In the above example an additional problem occurs: The right-hand side in the system of inequalities describing the polyhedron \( Q_\eta(x) \) is always an element of the affine subspace \( \{(u_1, u_2)^T \in \mathbb{R}^2 \mid u_1 = \eta \} \). That is due to the fact that the rows of \((DT - C)\) are not linearly independent and hence \((DT - C)\mathbb{R}^{m_1} \) is a subspace of \( \mathbb{R}^N \) whose dimension is at most \( N - 1 \). Because of (L1), there are no implicit equations among the inequalities describing \( Q_\eta(x_0) \). Therefore, each inequality is either redundant or corresponds to a facet of \( Q_\eta(x_0) \). In view of condition 1, we wish to be able to move all facets outward simultaneously, i.e. to pick a point \( \hat{x} \in X \) close to a given \( x \in X \) such that for a sufficiently small constant \( \sigma > 0 \) we have \( B_\sigma(0) \subseteq Q_\eta(x) \subseteq Q_\eta(\hat{x}) \). Therefore, we introduce the following assumptions:

(G2) \((DT - C)\mathbb{R}^{m_1} \cap \mathbb{R}^N_0 \neq \emptyset\).

(L2) \( x_0 \in \text{int} \ X \), i.e. there exists a constant \( \rho > 0 \) such that \( B_\rho(x_0) \subseteq X \).
Remark 4. \((L2)\) can only be fulfilled if the polyhedron \(X\) has full dimension, which leads to an additional global assumption.

Remark 5. \((G2)\) holds true if and only if the only solution to \(y^T(C - DT) = 0, y \geq 0\) is \(y = 0\) (see [20]).

Remark 6. If \((G2)\) is fulfilled, there exists a constant \(l > 0\) and a vector \(x_+ \in B_1(0) \subset \mathbb{R}^{m_1}\) such that
\[
(DT - C)x_+ \geq l \mathbf{1}.
\]
Furthermore, because of \((G1)\) and \((L1)\) there exists a constant \(\sigma > 0\) depending only on \(d_1, \ldots, d_N\) such that
\[
B_\sigma(0) \cup Q_\eta(x) \subseteq Q_\eta(x + x_+).
\]

Remark 7. We have \((DT - C)x > 0\) if and only if \(\min_{i=1,\ldots,N} p_i^T x > c^T x\), where \(p_i^T\) denotes the \(i\)-th row of \(DT\). If \(\|c\| > \max_{i=1,\ldots,N} \|p_i\|\), we can choose \(x_+ = -\frac{c}{\|c\|}\), because for every \(i \in \{1, \ldots, N\}\) we then have
\[
p_i^T(-c) \geq -\|p_i\|\|c\| > -\|c\|^2 = c^T(-c)
\]
and hence \((DT - C)(-c) > 0\).

So far, none of our assumptions involves the Borel measure \(\mu\) induced by the random variable \(z(\omega)\). The following paragraph shows that such additional assumptions are needed.

Finite discrete case: Consider the case that the random variable \(z\) has finitely many realizations \(z_1, \ldots, z_s \in \mathbb{R}^s\) with probabilities \(\pi_1, \ldots, \pi_s > 0\).
Assume \((A1)\), \((A2)\), \((G1)\) and let \(x_0 \in X\) be a point such that \(F_\eta(x_0) \geq 0\) and \((L1)\) is fulfilled. Without loss of generality we may assume \(z_1, \ldots, z_R \in \bar{Q}_\eta(x_0)\) and \(z_{R+1}, \ldots, z_S \notin \bar{Q}_\eta(x_0)\).
If \(\mu[Q_\eta(x_0)] = 1\), condition 1 cannot be fulfilled, since for \(x = x_0\) the left-hand side of (2.2) is \(\leq 1\), while the right-hand side is \(\geq 1\). We therefore assume \(\mu[Q_\eta(x_0)] < 1\) and hence \(R < S\). Since \(Q_\eta(x_0)\) is closed, we then have
\[
d := \min_{i=R+1,\ldots,S} \text{dist}_{PS}(z_i, Q_\eta(x_0)) > 0.
\]
According to remark 3, there exists a constant \(\epsilon > 0\) such that (2.3) holds true. That allows us to apply Hoffman’s theorem (see [23]): For arbitrary \(x \in B_s(x_0)\) we have
\[
\text{dist}_H(Q_\eta(x_0), Q_\eta(x)) \leq \beta \|DT - C\|_{\mathcal{L}(\mathbb{R}^{m_1}, \mathbb{R}^N)} \|x_0 - x\|,
\]
where \(\beta > 0\) is a constant not depending on \(x\). For sharp estimates of that constant we refer to [42]. Since \(\|DT - C\|_{\mathcal{L}(\mathbb{R}^{m_1}, \mathbb{R}^N)} = 0\) would imply \(Q_\eta(x_1) = Q_\eta(x_2)\) for all \(x_1, x_2 \in \mathbb{R}^{m_1}\), condition 1 could not be fulfilled. We therefore assume \(\|DT - C\|_{\mathcal{L}(\mathbb{R}^{m_1}, \mathbb{R}^N)} > 0\) and set
\[
\hat{\epsilon} := \min\{\epsilon, \frac{d}{2 \beta \|DT - C\|_{\mathcal{L}(\mathbb{R}^{m_1}, \mathbb{R}^N)}^2}\}.
\]
For every $x \in B_{\epsilon}(x_0)$, (2.6) then implies $\text{dist}_H(Q_{\eta}(x_0), Q_\eta(x)) \leq \frac{d}{2}$ and hence

$$\min_{i=R+1, \ldots, S} \text{dist}_{P_S}(z_i, Q_\eta(x)) \geq \frac{d}{2}.$$ 

Consequently $\mu(Q_{\eta}(x_0)] \geq \mu(Q_\eta(x)] \, \forall x \in B_{\epsilon}(x_0)$.

**Conclusion:** If the random variable $z$ has finitely many realizations and we assume (A1), (A2), (G1) and (L1), condition 1 cannot be fulfilled.

We now turn to the case where the random variable $z$ is continuous and introduce the following assumptions:

(G3) $\mu \in \mathcal{P}(\mathbb{R}^s)$ is absolutely continuous with respect to the Lebesgue measure, i.e. $\mu$ has a density $f_\mu$.

(G4) $\text{supp}(\mu)$ is compact and there exists a constant $v > 0$ such that $f_\mu \geq v$ $\mu$–almost everywhere on $\text{supp}(\mu)$.

**Remark 8.** (G4) ensures that for arbitrary $A, B \in \mathcal{B}(\mathbb{R}^s)$ with $A \subseteq B$ the inequality

$$\mu[B] \geq \mu[A] + v \lambda_s[(B \setminus A) \cap \text{supp}(\mu)]$$

holds true, which will prove useful in the context of condition 1. The mentioned condition can only be fulfilled if

$$\mu(Q_{\eta}(x_0)] < 1 \iff \mu[\text{supp}(\mu) \setminus Q_\eta(x_0)] > 0.$$ \hspace{1cm} (2.7)

Note that since $\text{supp}(\mu), Q_{\eta}(x_0)$ are Borel sets, so is $\text{supp}(\mu) \setminus Q_{\eta}(x_0)$. An additional assumption that guarantees (2.7) to hold true will be presented later on.

**Remark 9.** Assume (A1), (A2), (G1), (G3), (G4) and let $x_0 \in X$ be a point such that $F_\eta(x_0) \geq 0$ and (L1) is fulfilled. If

$$\inf_{z \in Q_{\eta}(x_0)} \text{dist}_{P_S}(z, \text{supp}(\mu)) > 0,$$

the argument used in the finite discrete case can be repeated to show that condition 1 cannot be fulfilled.

We additionally assume that

(L3) $\text{bd } Q_{\eta}(x_0) \cap \text{int supp}(\mu) \neq \emptyset$.

**Remark 10.** As part of the proof of the subsequent theorem 2.5 we will verify that (G4) and (L3) together imply (2.7) to hold.

We are now in the position to formulate our main result regarding verifiable sufficient conditions for the growth $F_{\eta}$, with fixed $\eta$, in the sense of [22], cf. Definition 2.1. For the benefit of the reader we recall the relevant assumptions made so far:
where $\text{relint}$ denotes the relative interior. Since (L1) holds true, the dimension of $F$ is $(s-1)$. 

**Step 1:** We first focus on the point $(L3)$ implies that there exists a facet $F \in P$ of a single facet of $Q$ of a single facet of $Q$

**Proof.** The proof proceeds by establishing condition 1, cf. (2.2), to hold. This is accomplished by a number of (sub-)steps each of them, for the benefit of the reader, beginning with a short introductory text. We show that condition 1 is fulfilled at $x_0$ and use lemma 2.4 to conclude that $F_\eta$ has to be (s-1), (G1) implies that there exist constants $\rho > 0$ such that $B_\rho(x_0) \subseteq X$.

**Step 1:** We first focus on the point $x_0$: condition 1 can only be fulfilled if (2.2) is fulfilled for $x = x_0$, which requires (2.7) to hold true. We show that supp($\mu$) \supset Q_\eta(x_0) contains a ball, whose projection on the set Q_\eta(x_0) is contained in the relative interior of a single facet of $Q_\eta(x_0)$.

(L3) implies that there exists a facet $F_{x_0}$ of $Q_\eta(x_0)$ such that

$$\text{relint } F_{x_0} \cap \text{int supp}(\mu) \neq \emptyset,$$

where relint denotes the relative interior. Since (L1) holds true, the dimension of $F_{x_0}$ is $(s-1)$. $F_{x_0}$ can be represented as the intersection of $Q_\eta(x_0)$ and an affine subspace $U_{F_{x_0}} \subseteq \mathbb{R}^s$. Without loss of generality we may assume that

$$U_{F_{x_0}} = \{ z \in \mathbb{R}^s \mid d_i^T z = b_i(x_0), \forall i \in \{1, \ldots, R\} \},$$

$$d_i^T z < b_i(x_0) \forall i \in \{R + 1, \ldots, N\} \forall z \in \text{relint } F_{x_0}. $$

Since the dimension of $U_{F_{x_0}}$ has to be $(s-1)$, (G1) implies that there exist constants $\alpha_2, \ldots, \alpha_R \in \mathbb{R}_{>0}$ such that

$$d_i = \alpha_i d_1, \quad b_i(x_0) = \alpha_i b_1(x_0) \quad \forall i \in \{2, \ldots, R\}.$$

Again because of (L1) we have $\alpha_2, \ldots, \alpha_R \in \mathbb{R}_{>0}$. These constants will be of importance in step 2 of this proof. (2.8) implies that there exists a point $z_0 \in F_{x_0}$ and a constant $\xi > 0$ such that

$$B_\xi(z_0) \cap U_{F_{x_0}} \subseteq F_{x_0} \quad \text{and} \quad B_\xi(z_0) \subseteq \text{int supp}(\mu).$$
Denote by \( N_{Q_\eta(x_0)}(z_0) \) the normal cone to \( Q_\eta(x_0) \) at \( z_0 \). We have
\[
N_{Q_\eta(x_0)}(z_0) = \{ \gamma \in \mathbb{R}^s \mid \langle \gamma, z - z_0 \rangle \leq 0 \ \forall \ z \in Q_\eta(x_0) \} = \text{cone}\{d_1\}.
\]
Setting \( d := \frac{1}{\|d_1\|}d_1 \in N_{Q_\eta(x_0)}(z_0) \), we have dist\(_{PS}(z_0 + \frac{\xi}{2} d, Q_\eta(x_0)) = \frac{\xi}{2} \). In addition, we conclude from (2.12) that
\[
B_{\frac{\xi}{2}}(z_0 + \frac{\xi}{2} d) \subseteq \text{supp}(\mu) \quad (2.13)
\]
\[
\Rightarrow B_{\frac{\xi}{2}}(z_0 + \frac{\xi}{2} d) \subseteq \text{supp}(\mu) \setminus Q_\eta(x_0). \quad (2.14)
\]

**Step 2:** Next, we show that for every \( x \) in a sufficiently small neighborhood of \( x_0 \) the polyhedron \( Q_\eta(x) \) has properties similar to (2.12) and (2.14) with a radius \( \xi \) that can be chosen independent on \( x \). That allows us to use the same argument to show that (2.2) holds true for \( x = x_0 \) and (2.2) holds true for arbitrary \( x \) in a sufficiently small neighborhood of \( x_0 \) with constants that do not depend on \( x \).

**Step 2.1:** First, we show that \( Q_\eta(x) \) has a facet parallel to \( F_{x_0} \). The proof includes some technicalities due to the fact that there might be redundant equations in the description of \( F_{x_0} \) (see (2.9), (2.11) with \( R > 1 \)).

Denote by \( d_{j,1}, \ldots, d_{j,s} \), the entries of \( d_j \), \( j = 1, \ldots, R \). Because of (G1), there exist indices \( i(1), \ldots, i(R) \in \{1, \ldots, s\} \) such that \( d_{j,i(j)} \neq 0 \) for every \( j \in \{1, \ldots, R\} \). With respect to the convention \( \min_{x \in \eta} x := +\infty \), which is important if \( R = N \), set
\[
g := \min_{j=R+1,\ldots,N} b_j(x_0) - d_j^T x_0 \quad (2.15)
\]
\[
d := \max\{|d_1|, \ldots, |d_N|, 1\} \geq \max\{|d_{i,1(i)}|, \ldots, |d_{R,i(R)}|\},
\]
\[
d \triangleq \min\{|d_{i,1(i)}|, \ldots, |d_{R,i(R)}|, 1\} \text{ and } r := \frac{g d}{2 \|DT - C\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^N)}}.
\]

Note that \( r \) does neither depend on \( x \) nor on \( \eta \). Now consider any \( x \in B_r(x_0) \). With the above choice of \( r \) we have in particular
\[
|b_j(x_0) - b_j(x)| \leq \|DT - C\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^N)} \|x_0 - x\| \leq \frac{g d}{2 d} \leq \frac{g}{2} \quad (2.16)
\]
for arbitrary \( j \in \{1, \ldots, N\} \). Set \( \alpha_1 := 1 \) and
\[
\sigma := \min_{j=1,\ldots,R} \frac{b_j(x)}{\alpha_j}. \quad (2.17)
\]
Without loss of generality we may assume
\[
\sigma = \frac{b_1(x)}{\alpha_1} = b_1(x). \quad (2.18)
\]
Consequently, the inequalities with row indices \( 2, \ldots, R \) are redundant in the description of \( Q_\eta(x) \), as for arbitrary \( j \in \{2, \ldots, R\} \) by \( d_j^T z \leq b_1(x) \) we have
\[
d_j^T z = \alpha_j d_j^T z \leq \alpha_j b_1(x) \leq b_j(x). \quad (2.18)
\]
Because of (2.9) these inequalities are redundant in the description of \( Q_\eta(x_0) \) as well. We delete the redundant inequalities from both descriptions. Set

\[
    z := z_0 - \frac{1}{d_{1, i(1)}} (b_1(x_0) - b_1(x)) e_{i(1)},
\]

where \( e_{i(1)} \in \mathbb{R}^s \) denotes the appropriate unit vector. We have

\[
    d_1^T z = d_1^T z_0 - b_1(x_0) + b_1(x) = b_1(x),
\]

\[
    \|z - z_0\| \leq \frac{g}{2d}. \tag{2.19}
\]

For \( j \in \{ R + 1, \ldots, N \} \) we conclude

\[
    d_j^T z = d_j^T z_0 + d_j^T (z - z_0) \leq d_j^T z_0 + \|d_j\| \|z - z_0\| \leq d_j^T z_0 + \frac{g}{2} \leq \frac{g}{2} \leq b_j(x).
\]

Hence, \( z \in Q_\eta(x) \cap \{ \bar{z} \in \mathbb{R}^s \mid d_1^T \bar{z} = b_1(x) \} \).

So far we have shown that for any \( x \in B_r(x_0) \) there exists an index \( j \in \{1, \ldots, R\} \), such that

\[
    \mathcal{F}_x := Q_\eta(x) \cap U_{\mathcal{F}_x} \neq \emptyset, \tag{2.21}
\]

where \( U_{\mathcal{F}_x} := \{ \bar{z} \in \mathbb{R}^s \mid d_1^T \bar{z} = b_1(x) \} \). If more than one choice for \( j \) is valid, i.e. the minimization problem in (2.17) has more than one solution, the affine subspace \( U_{\mathcal{F}_x} \) is invariant under the choice of \( j \).

**Step 2.2:** Next, we show that relint \( \mathcal{F}_x \) contains a \((s-1)\)-dimensional ball, whose radius does not depend on \( x \). The proof is done by applying Hoffman’s theorem and considering the projection of \( \mathcal{F}_x \) on the affine subspace \( U_{\mathcal{F}_x} \).

By (2.21), Hoffmann’s theorem is applicable and yields

\[
    \text{dist}_H(\mathcal{F}_{x_0}, \mathcal{F}_x) \leq \beta_j \| \begin{pmatrix} b_1(x_0) \\ b_{R+1}(x_0) \\ \vdots \\ b_N(x_0) \end{pmatrix} - \begin{pmatrix} b_1(x) \\ b_{R+1}(x) \\ \vdots \\ b_N(x) \end{pmatrix} \| \leq \beta_j \|b(x_0) - b(x)\| \leq \beta_j \|DT - C\|_{\mathcal{L}(\mathbb{R}^{m \times n}, \mathbb{R}^n)} \|x_0 - x\| \leq \beta_j \|DT - C\|_{\mathcal{L}(\mathbb{R}^{m \times n}, \mathbb{R}^n)} r,
\]

where the constant \( \beta_j > 0 \) may depend on \( j \). Setting

\[
    \beta := \max\{\beta_1, \ldots, \beta_R, 1\} \|DT - C\|_{\mathcal{L}(\mathbb{R}^{m \times n}, \mathbb{R}^n)},
\]

we get \( \text{dist}_H(\mathcal{F}_{x_0}, \mathcal{F}_x) \leq \beta r \). We now additionally demand that

\[
    r \leq \frac{\xi}{4\beta} \tag{2.22}
\]

and claim that there exists a point \( \bar{z} \in \mathcal{F}_x \) such that

\[
    B_{\bar{z}}(\bar{z}) \cap U_{\mathcal{F}_x} \subseteq \mathcal{F}_x \text{ and } B_{\bar{z}}(\bar{z}) \subseteq \text{int supp}(\mu).
\]
Denote by \( \text{proj}_{U_{F_{x_0}}} (F_x) \) the projection of \( F_x \) on the affine subspace \( U_{F_{x_0}} \). The affine subspaces \( U_{F_{x_0}} \) and \( U_{F_{x}} \) being parallel, \( \text{proj}_{U_{F_{x_0}}} : U_{F_{x_0}} \to U_{F_{x}} \) is an isometry. We have

\[
\text{dist}_H(F_{x_0}, \text{proj}_{U_{F_{x_0}}} (F_x)) \leq \beta r \leq \frac{\xi}{4},
\]

hence there exists a point \( z \in \text{proj}_{U_{F_{x_0}}} (F_x) \) with

\[
\|z - z_0\| \leq \frac{\xi}{4}.
\]

Suppose there exists a point \( \tilde{z} \in (B_{\frac{\xi}{2}}(z) \cap U_{F_{x_0}}) \setminus \text{proj}_{U_{F_{x_0}}} (F_x) \). Since \( F_x \) is a polyhedron, the same holds true for \( \text{proj}_{U_{F_{x_0}}} (F_x) \) and the line segment connecting \( z \) and \( \tilde{z} \) must contain a point \( \bar{z} \in \text{bd} \text{proj}_{U_{F_{x_0}}} (F_x) \). Pick any direction

\[
\bar{d} \in N_{\text{proj}_{U_{F_{x_0}}} (F_x)}(\bar{z}) \cap U_{F_{x_0}}, \quad \|\bar{d}\| = 1.
\]

Since

\[
B_{\frac{\xi}{2}}(\bar{z}) \cap U_{F_{x_0}} \subset B_{\frac{\xi}{2}}(z) \cap U_{F_{x_0}} \subset B_{\frac{\xi}{2}}(z_0) \cap U_{F_{x_0}} \subset F_{x_0},
\]

we have \( \bar{z} + \frac{\xi}{2} \bar{d} \in F_{x} \setminus \text{proj}_{U_{F_{x_0}}} (F_x) \). Because of (2.25) we have

\[
\text{dist}_{PS}(\bar{z} + \frac{\xi}{2} \bar{d}, \text{proj}_{U_{F_{x_0}}} (F_x)) = \frac{\xi}{2}
\]

and therefore \( \text{dist}_H(F_{x_0}, \text{proj}_{U_{F_{x_0}}} (F_x)) \geq \frac{\xi}{2} \), contradicting (2.23). Hence it holds \( B_{\frac{\xi}{2}}(z) \cap U_{F_{x_0}} \subset \text{proj}_{U_{F_{x_0}}} (F_x) \). Denote by \( \bar{z} \in F_x \) the unique point of \( F_x \) satisfying \( \text{proj}_{U_{F_{x_0}}} (\bar{z}) = z \). Then

\[
B_{\frac{\xi}{2}}(\bar{z}) \cap U_{F_x} \subset F_x
\]

and together with

\[
\text{dist}_H(U_{F_{x_0}}, U_{F_x}) = |b_j(x_0) - b_j(x)| \leq \|DT - C\|_{\mathcal{L}(\mathbb{R}^m, \mathbb{R}^N)} r \leq \frac{\xi}{4},
\]

(2.24) implies

\[
\|z_0 - \bar{z}\| \leq \frac{\xi}{2}.
\]

Therefore,

\[
B_{\frac{\xi}{4}}(\bar{z}) \subset B_{\frac{\xi}{2}}(z_0) \subset \text{int supp}(\mu).
\]

**Step 2.3:** It remains to show that \( \text{supp}(\mu) \setminus Q_0(x) \) contains a ball with a property similar to (2.14).
Since $F_x$ and $F_{x_0}$ are parallel, we have $N_{Q_\eta(x_0)}(z_0) = N_{Q_\eta(x)}(\hat{z}) = \text{cone}\{d_1\}$. Setting $d := \frac{1}{\|d_1\|} d_1$ we observe $B_{\xi}^{\frac{1}{4}}(\hat{z} + \frac{\xi}{4} d) \cap Q_\eta(x) = \emptyset$. Now (2.27) implies
\[ \|z_0 - (\hat{z} + \frac{\xi}{4} d)\| \leq \frac{3\xi}{4} \]
and, because of (2.12), we have
\[ B_{\frac{\xi}{4}}(\hat{z} + \frac{\xi}{4} d) \subseteq \text{supp}(\mu). \] (2.29)
Hence,
\[ B_{\frac{\xi}{4}}(\hat{z} + \frac{\xi}{4} d) \subseteq \text{supp}(\mu) \setminus Q_\eta(x). \] (2.30)

**Step 3:** We show that (2.2) holds true in $x$ by constructing a point $\hat{x} \in B_v(x) \cap X$ that has the desired property. Moreover, we show that the constant $p > 0$ can be chosen independent on $x$ and $\hat{x}$.

**Step 3.1:** We apply an appropriate transformation of coordinates to simplify the notation.

Denote by $B_{\frac{\xi}{s}}(\hat{z})$ the $\|\cdot\|_\infty$-ball with radius $\frac{\xi}{s}$ and center $\hat{z}$. Since $s \in \mathbb{N}$, we have $\|a\|_\infty \leq \sqrt{s}\|a\| \leq s\|a\| \forall \ a \in \mathbb{R}^s$, implying
\[ B_{\frac{\xi}{s}}(\hat{z}) \subseteq B_{\frac{\xi}{s\sqrt{s}}}^{\infty}(\hat{z}) \subseteq B_{\frac{\xi}{s}}(\hat{z}) \]
and therefore
\[ B_{\frac{\xi}{\sqrt{s}}}^{\infty}(\hat{z}) \cap U_{F_x} \subseteq B_{\frac{\xi}{4}}(\hat{z}) \cap U_{F_x} \subseteq F_x. \] (2.31)

Set $U_1 := \{z \in \mathbb{R}^s \mid e_1^T z = 0\}$. After applying an appropriate translation and rotation to $\mathbb{R}^s$, we may assume that $U_{F_x} = U_1$, $\hat{z} = 0$ and
\[ Q_\eta(x) \cap \{z \in \mathbb{R}^s \mid e_1^T z > 0\} = \emptyset. \] (2.32)
Because of (2.30) and (2.32) we have
\[ B_{\frac{\xi}{s}}^{\infty}(\frac{\xi}{4} e_1) \cap \left(\frac{\xi}{4} e_1 + U_1\right) \subseteq \text{supp}(\mu) \setminus Q_\eta(x). \]

**Step 3.2:** We construct $\hat{x}$ in a way guaranteeing that $Q_\eta(x) \subset Q_\eta(\hat{x})$ and that $(\text{supp}(\mu) \cap Q_\eta(\hat{x})) \setminus Q_\eta(x)$ contains a cartesian product of $s$ nontrivial intervals. We show that $s-1$ of these intervals can be chosen independent of $x$ and $\hat{x}$.

Since (L2) holds true, we can additionally choose $r > 0$ so small that $B_r(x_0) \subset X$ and therefore
\[ B_{\frac{r}{4}}(x) \subset X \forall x \in B_r(x_0). \] (2.33)
Now fix an arbitrary $\epsilon > 0$. Without loss of generality we may assume
\[
d_i^T z = b_i(x) \quad \forall i \in \{1, \ldots, S\} \quad \forall z \in F_x, \\
d_i^T z < b_i(x) \quad \forall i \in \{S + 1, \ldots, N\} \quad \forall z \in \text{relint } F_x.
\]
By (G2), (2.4) yields $(DT-C)(x+\kappa x_+) \geq (DT-C)x+\kappa l_1$ for arbitrary $\kappa > 0$. Note that the constant $l > 0$ does neither depend on $x$ nor on $S$. Set $\hat{x} := \hat{x}(\kappa) := x + \kappa x_+$, then by (2.5) we have $B_{\kappa}(0) \oplus Q_{\eta}(x) \subseteq Q_{\eta}(\hat{x})$, where the constant $\sigma > 0$ does only depend on $d_1, \ldots, d_N$. In addition,
\[
\|\hat{x} - x\| = \kappa. \tag{2.34}
\]
Now choose $\kappa > 0$ so small that $\kappa \leq \min\{\epsilon, \frac{\epsilon}{4}\}$. Because of (2.33) and (2.34) that yields $\hat{x} \in B_{\kappa}(x) \cap X$. (2.31) and (2.28) imply
\[
B_{\infty}^\infty(0) \cap U_1 \subseteq F_x \quad \text{and} \quad B_{\infty}^\infty(0) \subseteq \text{supp}\mu. \tag{2.35}
\]
Furthermore, there exists a constant $L > 0$ depending on the rows of $(DT-C)$ and on $x_+$ but not on $x$ or $\eta$, such that for every $\kappa < L$ we have
\[
B_{\infty}^\infty\left(\hat{u} \kappa e_1 \right) \cap (\hat{u} \kappa e_1 + U_1) \subseteq Q_{\eta}(\hat{x}), \tag{2.36}
\]
where $\hat{u} > 0$ is a constant that only depends on the rows of $D$ defining the facet $F_x$, i.e. on $d_1, \ldots, d_S$. Hence, no more than $N$ different constants $\hat{u}$ can arise and their minimum $u$ is strictly positive and independent of $x$ and $\eta$. This observation will be of importance when we are aiming for uniform local linear growth for all $\eta \in A \subseteq \mathbb{R}$ in section 3. Without loss of generality we assume that $u = 1$.
\[
B_{\infty}^\infty\left(\sum \frac{\epsilon}{4} e_1 \right) \subseteq \text{supp}\mu \tag{2.37}
\]
and together with (2.35), (2.36) and the convexity of the polyhedron $Q_{\eta}(\hat{x})$ we can conclude
\[
[0,\kappa] \times P_1(B_{\infty}^\infty(0)) \subseteq Q_{\eta}(\hat{x}) \cap \text{supp}\mu, \tag{2.38}
\]
where $P_1(B_{\infty}^\infty(0))$ denotes the set
\[
P_1(B_{\infty}^\infty(0)) := \{z \in \mathbb{R}^{s-1} \mid (0, z)^T \in \text{proj}_{\{0\}} \times \mathbb{R}^{s-1}(B_{\infty}^\infty(0))\}.
\]
Because of (2.32) we have
\[
[0,\kappa] \times P_1(B_{\infty}^\infty(0)) \subseteq (Q_{\eta}(\hat{x}) \cap \text{supp}\mu) \setminus Q_{\eta}(x). \tag{2.39}
\]
**Step 3.3:** It finally remains to show that (2.38) implies (2.2).

Note that the set on the left-hand side of (2.38) is a cartesian product of intervals. Denote by $\lambda_{s-1}$ the $(s-1)$-dimensional Lebesgue measure. We obtain
\[
\mu[Q_{\eta}(\hat{x})] \geq \mu[Q_{\eta}(x)] + \mu[[0,\kappa] \times P_1(B_{\infty}^\infty(0))]
\]
\[
\overset{(G4)}{=} \mu[Q_{\eta}(x)] + v \lambda_{s-1}[[0,\kappa] \times P_1(B_{\infty}^\infty(0))]
\]
\[
= \mu[Q_{\eta}(x)] + v \kappa \lambda_{s-1}[P_1(B_{\infty}^\infty(0))]
\]
\[
\overset{(2.34)}{=} \mu[Q_{\eta}(x)] + v \|x - \hat{x}\| \lambda_{s-1}[P_1(B_{\infty}^\infty(0))].
\]
Note that neither the constant $v > 0$ nor $\lambda_{s-1}[P_{l}(B_{\infty}^{-1}(0))]$ depend on $x$, therefore condition 1 is fulfilled with $p := \frac{v}{2} \lambda_{s-1}[B_{\infty}^{-1}(0)]$. Since $F_{\eta}(x_0) \geq 0$ and $F_{\eta}$ is upper semicontinuous on $\mathbb{R}^{m_1}$ according to lemma 2.4, $F_{\eta}$ is growing at $x_0$ with respect to $X$. □

3. Sufficient conditions for local linear growth uniformly in $\eta$.

Formula (2.1) yields a representation of the feasible set $C(\mu)$ via a system with uncountably many inequalities. We consider a relaxation of problem $P(\mu)$ via replacing $\eta \in \mathbb{R}$ in (2.1) by $\eta \in A$, with some subset $A \subseteq \mathbb{R}$. This leads to $F_A : \mathbb{R}^{m_1} \to [-1, \infty)$, $F_A(x) := \inf_{\eta \in A} F_{\eta}(x)$. Furthermore, with some $l \in \mathbb{R}$, we will study the set-valued mapping $C_{A,l} : \mathcal{P}(\mathbb{R}^n) \to 2^{\mathbb{R}^{m_1}}$ given by

$$C_{A,l}(\mu) := \{x \in X \mid F_A(x) \geq l\}.$$ 

For arbitrary $\sigma \in \mathcal{P}(\mathbb{R}^n)$ we have $C(\sigma) = C_{A,0}(\sigma)$ and set $C_A(\sigma) := C_{A,0}(\sigma)$.

So far, we have considered $\eta \in \mathbb{R}$ to be fixed and shown that under the assumptions of theorem 2.5 condition 1 holds true at $x_0 \in \mathbb{R}^{m_1}$. Denote by $r(x_0, \eta)$ and $p(x_0, \eta)$ the associated constants from condition 1.

In the present section we derive verifiable sufficient conditions for $F_A$ to grow at $x_0$ with respect to $X$.

**Corollary 3.1.** Let the assumptions of theorem 2.5 and $B_{r(x_0, \eta)}(x_0) \subset X$ hold true for $(x_0, \eta) \in \mathbb{R}^{m_1} \times \mathbb{R}$ and assume that there exists a point $x_1 \in \mathbb{R}^{m_1}$ such that $(DT - C)x_1 = 1$. Furthermore, assume

$$F_{\eta, r(x_0, \eta)}(x_0) \geq 0,$$  

and set $A_{\eta} := [\eta - \frac{r(x_0, \eta)}{2\|x_1\|}, \eta + \frac{r(x_0, \eta)}{2\|x_1\|}]$. Then $F_{A_{\eta}}$ is growing at $x_0$ with respect to $X$.

**Proof.** Fix an arbitrary $\eta' \in A_{\eta}$ and set

$$x_{\eta'} := (\eta - \eta')x_1 \in \mathbb{R}^{m_1}. \quad (3.2)$$

Then we have $(DT - C)x_{\eta'} = (\eta - \eta')1$ and $\|x_{\eta'}\| \leq \frac{r(x_0, \eta)}{2 \|x_1\|}$. Consider an arbitrary point $x' \in B_{r(x_0, \eta)}(x_0) = B_{\frac{r(x_0, \eta)}{2}}(x_0)$ and set $x := x' - x_{\eta'}$. Observe $Q_{\eta'}(x') = Q_{\eta}(x)$ and

$$\|x - x'\| \leq \|x - x'\| + \|x' - x_0\| \leq \|x_{\eta'}\| + \frac{r(x_0, \eta)}{2} \leq r(x_0, \eta),$$

therefore in particular $x \in X$. Fix an arbitrary $\epsilon \in (0, \frac{r(x_0, \eta)}{2 \|x_1\|})$. According to the proof of theorem 2.5, there exists a point $\hat{x} \in B_{\epsilon}(x) \cap X$ such that

$$\mu[Q_{\eta'}(\hat{x})] > \mu[Q_{\eta}(x)] + p(x_0, \eta) \|\hat{x} - x\|. \quad (3.3)$$

Note that due to the construction in step 3.2 of the proof of theorem 2.5 we have $\hat{x} = x + \kappa x_+ = x' + x_{\eta'} + \kappa x_+$, where $\kappa > 0$ can be chosen independent of $x$. Set $\hat{x}' := \hat{x} + x_{\eta'} = x' + \kappa x_+$, then $\hat{x}'$ does not depend on $\eta'$ and we have $Q_{\eta'}(\hat{x}') = Q_{\eta}(\hat{x})$. Furthermore,

$$\|\hat{x}' - x'\| = \|\hat{x} - x\| \leq \epsilon \quad (3.4)$$
hold true. (3.4) implies
\[ \| \hat{x}' - x_0 \| \leq \| \hat{x}' - x' \| + \| x' - x_0 \| \leq \epsilon + \frac{r(x_0, \eta)}{2} \leq r(x_0, \eta) \]
and hence \( \hat{x}' \in X \). Using the last results, we rewrite (3.3) as
\[ \mu[Q_{\eta'}(\hat{x}')] > \mu[Q_{\eta'}(x')] + p(x_0, \eta) \| \hat{x}' - x' \| = \mu[Q_{\eta'}(x')] + p(x_0, \eta') \| \hat{x}' - x' \|. \]
Adding \(-\nu[\mathbb{R}_{\leq \eta'}]\) to both sides we conclude
\[ F_{\eta'}(\hat{x}') > F_{\eta'}(x') + p(x_0, \eta') \| \hat{x}' - x' \|. \]
Hence condition 1 is fulfilled at \((x_0, \eta')\) with constants \(r(x_0, \eta') = \frac{r(x_0, \eta)}{2}\) and \(p(x_0, \eta') = p(x_0, \eta)\). Since \( \hat{x}' \) does not depend on \(\eta'\), it holds true that
\[ F_{\eta}(\hat{x}') > F_{\eta}(x') + p(x_0, \eta) \| \hat{x}' - x' \| \quad \forall \eta \in A_{\eta}. \]
Hence, \( F_{A_{\eta}}(\hat{x}') > F_{A_{\eta}}(x') + p(x_0, \eta) \| \hat{x}' - x' \|\). Because of (3.1) we have
\( F_{\eta}(x_0) \geq F_{\eta} - \frac{r(x_0, \eta)}{2} > 0 \) for all \(\eta \in A_{\eta}\) and hence \( F_{A_{\eta}}(\hat{x}') > 0 \). Lemma 2.4 yields that \( F_{A_{\eta}} \) is the pointwise infimum over a family of upper semicontinuous functions.
Consequently, \( F_{A_{\eta}} \) itself is upper semicontinuous and hence growing at \(x_0\) with respect to \(X\).

**Corollary 3.2.** Set \( A_{\eta} := [\eta - \frac{r(x_0, \eta)}{2}, \eta + \frac{r(x_0, \eta)}{2}] \). Under the assumptions of corollary 3.1, \( F_{A_{\eta}} \) is growing at every \(x \in B_{\frac{r(x_0, \eta)}{2}}(x_0)\) with respect to \(X\).

**Proof.** Observe that \( F_{\eta} \) is growing at \(x \in B_{\frac{r(x_0, \eta)}{2}}(x_0)\) with respect to \(X\) and constants \(r(x, \eta) = \frac{r(x_0, \eta)}{2}, p(x, \eta) = \frac{p(x_0, \eta)}{2}\), then apply corollary 3.1.

There are two problems with the above corollaries:
- The mappings \( F_{A_{\eta}} \) and \( F_{A_{\eta}} \) depend on \(x_0\).
- Considering \( F_{A_{\eta}} \) instead of \( F_{\eta} \) changes the set of feasible points in general, i.e. \( C_{A_{\eta}} \subset C_{\eta} \). The same holds true for \( C_{A_{\eta}} \).

Therefore, we take a different approach and develop additional assumptions that allow us to avoid the mentioned problems. Because of (G4), for every \(x \in \mathbb{R}^{m_1}\) there exists a constant \(\gamma_0 \in \mathbb{R}\) such that \(Q_{\eta}(x) \cap \text{supp}(\mu) = \emptyset\) for all \(\eta \leq \gamma_0\). Consequently, \( F_A(x) \leq 0 \) for all \(x \in \mathbb{R}^{m_1}\) for every set \(A \subseteq \mathbb{R}\) that is not bounded from below and therefore \( C_A(\mu) = \{x \in X \mid F_A(x) = 0\} \). Hence, the condition 1 cannot be fulfilled at any \(x_0 \in C_A(\mu)\). In addition, if the distribution function of the benchmark profile \(d\) is strictly positive on \(A\), e.g. when \(d\) is normally distributed, we have \( F_A(x) < 0 \) for all \(x \in \mathbb{R}^{m_1}\) and hence \( C_A(\mu) = \emptyset \).

We restrict ourselves to a special class of benchmark profiles and introduce the following additional assumption:

**(G5)** The benchmark profile \(d\) is discrete with finitely many realizations \(a_1 < \ldots < a_k\) and probabilities \(\pi_1, \ldots, \pi_k > 0\).

We then have
\[
\nu[\mathbb{R}_{\leq \eta}] = \begin{cases} 
0, & \text{if } \eta < a_1 \\
\sum_{j=1}^{i} \pi_j, & \text{if } a_i \leq \eta < a_{i+1}, \; i = 1, \ldots, k - 1 \\
1, & \text{if } a_k \leq \eta
\end{cases}
\]
Remark 11. If the first part of (L3**) yields while the second part implies

\[ F_{\eta}(x) = \min \{0, \min_{i=1, \ldots, k} \mu(Q_{a_i}(x)) - \sum_{j=1}^{i} \pi_j \} \]

Furthermore,

\[ F_{[a_1, \infty)}(x) = \min_{i=1, \ldots, k} \mu(Q_{a_i}(x)) - \sum_{j=1}^{i} \pi_j = F_{[a_1, \ldots, a_k]}(x). \]

It holds true that \( F_{\eta}(x) \geq 0 \iff F_{\eta}(x) = 0 \iff F_{[a_1, \ldots, a_k]}(x) \geq 0 \) and hence \( C_{[a_1, \ldots, a_k]}(\mu) = C(\mu) \). Therefore, considering \( F_{[a_1, \ldots, a_k]} \) instead of \( F_{\eta} \) can be seen as changing to a different representation of the same feasible set.

We replace the local assumptions made in section 2 with a set of assumptions that do not involve \( \eta \):

\begin{enumerate}
\item[(L1*)] The polyhedron \( Q_{a_i}(x_0) \) has full dimension
\item[(L2*)] \( x_0 \in \text{int} X \)
\item[(L3*)] \( \text{bd} \ Q_{a_i}(x_0) \cap \text{int supp}(\mu) \neq \emptyset \) for all \( i = 1, \ldots, k \)
\item[(L4*)] \( F_{[a_1, \ldots, a_k]}(x_0) \geq 0 \)
\end{enumerate}

Remark 11. If \( \text{supp}(\mu) \) is convex, assumption (L3*) can be weakened to read

\begin{enumerate}
\item[(L3**) \( \text{bd} \ Q_{a_i}(x_0) \cap \text{int supp}(\mu) \neq \emptyset \) and \( \text{bd} \ Q_{a_i}(x_0) \cap \text{int supp}(\mu) \neq \emptyset \).
\end{enumerate}

The first part of (L3**) yields

\[ Q_{a_i}(x_0) \cap \text{int supp}(\mu) \neq \emptyset \quad \forall i \in \{1, \ldots, k\}, \tag{3.5} \]

while the second part implies

\[ \text{int supp}(\mu) \setminus Q_{a_i}(x_0) \neq \emptyset \quad \forall i \in \{1, \ldots, k\}. \tag{3.6} \]

(3.5) and (3.6) together with the convexity of \( \text{supp}(\mu) \) yield

\[ \text{bd} \ Q_{a_i}(x_0) \cap \text{int supp}(\mu) \neq \emptyset \quad \forall i \in \{1, \ldots, k\}. \]

Hence, (L3**) implies (L3*).

The following theorem represents our main result on sufficient conditions for growth in the sense of [22].

Theorem 3.3. Assume that (A1), (A2), (G1) - (G5) are fulfilled and let \( x_0 \in X \) be a point such (L1*) - (L4*) hold true. Then \( F_{[a_1, \ldots, a_k]} \) is growing at \( x_0 \) with respect to \( X \).

Proof. Since \( Q_{a_1}(x_0) \subset \ldots \subset Q_{a_k}(x_0) \), (L1*) implies that \( Q_{a_i}(x_0) \) has full dimension for all \( i = 1, \ldots, k \). By (L2*), (L4*) we have \( x_0 \in C_{[a_1, \ldots, a_k]}(\mu) \). Hence, theorem 2.5 is applicable for \( F_{a_1}, \ldots, F_{a_k} \) and yields that there exist constants \( r_1, \ldots, r_k > 0 \) and \( p_1, \ldots, p_k > 0 \) such that for every \( i \in \{1, \ldots, k\} \), all \( x \in X \cap B_{r_i}(x_0) \) and \( \epsilon > 0 \) there exists a point \( \hat{x}_i \in X \cap B_{\epsilon}(x) \) with \( F_{a_i}(\hat{x}_i) > F_{a_i}(x) + p_i \|\hat{x}_i - x\| \). Set \( r = \min_{i=1, \ldots, k} r_i \)
and \( p = \min_{i=1,\ldots,k} p_i. \) Then for every \( x \in X \cap B_r(x_0) \) and \( \epsilon > 0 \) it is possible to choose \( \tilde{x}_1 = \ldots = \tilde{x}_k \) due to the construction in step 3.2 of the proof of theorem 2.5. Consequently, condition 1 is fulfilled for \( F_{(a_1,\ldots,a_k)} \) at \( x_0. \) Since by lemma 2.4 \( F_{(a_1,\ldots,a_k)} \) is the pointwise minimum of upper semicontinuous functions, \( F_{(a_1,\ldots,a_k)} \) itself is upper semicontinuous and hence growing at \( x_0 \) with respect to \( X. \)

**Remark 12.** Assuming (G5), the feasible set \( C[\mu] = C_{(a_1,\ldots,a_k)} [\mu] \) can be represented via a system of finitely many chance constraints. If \( \mu \) is r-concave for a constant \( r < 0, \) i.e. \( \mu^r[A] + (1 - \lambda)B \leq \lambda \mu^r[A] + (1 - \lambda)\mu^r[B] \) (with \( \mu^r[C] = \infty \) if \( \mu[C] = 0 \)) for all \( \lambda \in [0,1] \) and all convex Borel sets \( A, B \) such that \( \Lambda A + (1 - \lambda)B \) is also a Borel set, metric regularity can be verified without applying the characterization given by theorem 2.2: In that case, the feasible set \( C[\mu] = C_{(a_1,\ldots,a_k)} [\mu] \) is convex and the existence of a Slater point \( x_S \in X \) such that \( F_{(a_1,\ldots,a_k)}(x_S) > 0 \) implies that \( \Gamma_{F_{a,k}} \) is metrically regular at \( (x_0,0) \) for every \( x_0 \in C[\mu] \) (see the proof of Corollary 3.7 in [37]). Note that, in case \( z \) is uniformly distributed on a nonconvex set, \( \mu \) no longer needs to be r-concave: For example, if \( \text{supp}(\mu) = [0,1] \cup [3,4], \) considering \( A = [0,2], B = [3,4] \) and \( \lambda = \frac{1}{2} \) we get \( \mu^r[A] + (1 - \lambda)B = \mu^r([1,3]) = \infty, \) but \( \frac{1}{2} \mu^r[A] + \frac{1}{2} \mu^r[B] < \infty. \)

4. Implications.

We conclude the paper with our main results on stability and sensitivity of the class of stochastic programs with first-order dominance constraints addressed. These results hinge upon the sufficient growth conditions established in the previous sections, thus justifying the technical effort spent there.

**Theorem 4.1.** Let \( A \subseteq \mathbb{R} \) be a set and \( x_0 \in C_A(\mu) \) a point such that \( F_A \) is growing at \( x_0 \) with respect to \( X. \) Then the following statements hold true:

1. The associated set-valued mapping \( \Gamma_{F_A} : \mathbb{R}^m \to \mathbb{R}^2 \) given by

\[
\Gamma_{F_A}(x) = \begin{cases} 
\{F_A(x)\} \odot \mathbb{R}_{\geq 0}, & \text{if } x \in X \\
\emptyset, & \text{else}
\end{cases}
\]

is metrically regular at \( (x_0,0) \) with respect to \( X. \)

2. The set-valued mapping \( \Xi_A : \mathbb{R} \to 2^{\mathbb{R}^m} \) given by

\[
\Xi_A(l) := C_{A,l}(\mu) = \{x \in X \mid F_A(x) \geq l\}
\]

is pseudo-Lipschitzian at \( (0,x_0). \)

3. The mapping \( d_A : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}_{\geq 0} \) given by

\[
d_A(x,l) := \text{dist}_{PS}(x,C_{A,l}(\mu))
\]

is locally Lipschitzian at \( (x_0,0). \)

**Proof.** The first part is theorem 2.2. See theorem 1.5 in [32] for the second and theorem 2.3 in [34] for the third part.

Let \( A \subseteq \mathbb{R} \) be fixed and consider the problem

\[
\min \{g(x) \mid x \in C_A(\mu)\}. \tag{4.1}
\]
Given an open set $V \subseteq \mathbb{R}^{m_1}$ we set for each $\sigma \in \mathcal{P}(\mathbb{R}^s)$
\[
\varphi_V(\sigma) := \inf \{ g(x) \mid x \in C_A(\mu) \cap \text{cl } V \} \quad \text{and} \quad \Psi_V(\sigma) := \text{argmin} \{ g(x) \mid x \in C_A(\mu) \cap \text{cl } V \},
\]
where $\text{cl } V$ denotes the closure of $V$.

**Theorem 4.2.** Assume that
1. $Q$ is a CLM set for problem (4.1) with respect to an open and bounded set $V$, i.e. $Q = \Psi_V(\mu) \subset V$ and $Q$ is compact,
2. $g$ is locally Lipschitzian and
3. $\Gamma_F$ is metrically regular at $(x_0, 0)$ with respect to $X$ for every $x_0 \in Q$,
4. $B$ is a determining class of Borel sets in $\mathbb{R}^s$ such that $Q_\eta(x) \in B$ for all $\eta \in A$ and $x \in X$
Then there exist constants $L > 0$ and $\delta > 0$ such that the set-valued mapping $\Psi_V : (\mathcal{P}(\mathbb{R}^s), \text{dist}_B) \to 2^{\mathbb{R}^{m_1}}$ is upper semicontinuous at $\mu$, $\Psi_V(\sigma)$ is a CLM-set for
\[
\min \{ g(x) \mid x \in C_A(\sigma) \}
\]
with respect to $V$ and $|\varphi_V(\mu) - \varphi_V(\sigma)| \leq L \text{dist}_B(\mu, \sigma)$ holds whenever $\sigma \in \mathcal{P}(\mathbb{R}^s)$ and $\text{dist}_B(\mu, \sigma) \leq \delta$. Moreover, if in addition there exists a constant $c > 0$ such that
\[
g(x) \geq \varphi_V(\mu) + c \text{dist}_{PS}(x, \Psi_V(\mu))^2 \quad \forall x \in C_A(\mu) \cap V
\]
then $\Psi_V$ is Hausdorff-Hölder continuous with rate $\frac{1}{2}$, i.e.
\[
\sup_{x \in \Psi_V(\sigma)} \text{dist}_{PS}(x, \Psi_V(\mu)) \leq L \text{dist}_B(\mu, \sigma)^{\frac{1}{2}}
\]
whenever $\sigma \in \mathcal{P}(\mathbb{R}^s)$ and $\text{dist}_B(\mu, \sigma) \leq \delta$.

**Proof.** Combine theorem 4.1 with theorem 1 from [22].

**Remark 13.** If (A1), (A2) and (G5) hold true, $A$ is the set of realizations of the benchmark variable $d$ and $\mu$ is $r$-concave for a constant $r < 0$, the localization in first part of the implications of the previous theorem can be dropped. That means the boundedness of $V$ in the first assumption is not needed and the results hold true globally, i.e. for $\Psi = \Psi_{\mathbb{R}^m}$ and $\phi = \phi_{\mathbb{R}^m}$. That can be shown following the ideas in [21], where the proof is given for the case of a single chance constraint (see Theorem 3.1).

**Remark 14.** The result can easily be expanded to the metrizable topology induced by weak convergence on $\mathcal{P}(\mathbb{R}^s)$: If $B$ is a subclass of all convex Borel sets we have
\[
\mu[\text{bd } B] = 0 \quad \forall B \in B
\]
(see [25]). That implies
\[
\mu_n \overset{w}{\rightharpoonup} \sigma \in \mathcal{P}(\mathbb{R}^s) \implies \lim_{n \to \infty} \text{dist}_B(\mu_n, \mu) = 0
\]
and hence the above theorem holds true when $\mathcal{P}(\mathbb{R}^s)$ is equipped with the metrizable topology induced by weak convergence instead of the topology induced by $\text{dist}_B$. 

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Stability of stochastic dominance constraints


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